Convergence of Non-negative Terms of the Series

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Abstract

In this paper, some concepts of the sequences are applied. Firstly, some definitions and theorems about the convergence of the sequences and limits of the sequences are presented. Finally, some definitions and theorems about the convergence of the series, especially the non-negative terms of the series are discussed.

Keywords: Sequence, series and convergent of series.

1. Introduction

The Indian Mathematicians had interest in the study of series as early as the third century A.D. Their work on series continued till late fourteen century, but they never took a systematic and critical study. In Europe, it was during the sixth century, A.D. that the wider significance of the infinite series was realized. The English Mathematicians made important contributions to the study of infinite series. This paper will show about some concepts of the convergence of non-negative terms of the series with some definitions, theorems and applications with some examples.

2. Convergent Sequences

2.1 Definitions

A sequence $\{x_n\}$ of real numbers can be defined by a function which maps each natural number n into the real number x_n. A real number ℓ is a **limit of the sequence** $\{x_n\}$ if for each $\varepsilon > 0$, there is an integer N such that for all $n \ge N \Rightarrow |x_n - \ell| < \varepsilon$. It is easily proved that a sequence can have at most one limit, and denote this limit by $\lim x_n$ when it exists. In symbol $\ell = \lim\limits_{n\to\infty} x_n$ if $\varepsilon > 0$, $\exists$ N such that $n \ge N$ implies that $|x_n - \ell| < \varepsilon$.

Extend this limit of a sequence $\{x_n\}$ as $\lim\limits_{n\to\infty} x_n = \infty$ if given Δ, there is an integer N such that for all $n \ge N$ implies that $x_n > \Delta$. A sequence is called **convergence** if it has a limit. Convergence of sequence depends on whether or not a limit is a real number or an extended real number. It is more usual to use the restricted definition for convergence which requires a limit to be a real number. It is important to distinguish between the two concepts of a limit, explicitly, “converges to a real number” or “convergence” in the set of extended real numbers”.

In the case of a real number, ℓ is the limit of $\{x_n\}$ if given $\varepsilon > 0$, all but a finite number of terms of the sequence $\{x_n\}$ are within ε of ℓ, i.e., infinitely many terms of the sequence within ε of ℓ. In this case ℓ is a **cluster point** of the sequence $\{x_n\}$. ℓ is a cluster point of $\{x_n\}$ if, given $\varepsilon > 0$, and given an integer N, $\exists\ n \ge N$ such that $|x_n - \ell| < \varepsilon$. Thus if a sequence has a limit ℓ,

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then ℓ is a cluster point, but the converse is not usually true. Cluster points of a real sequence are also called accumulation or condensation points.

2.2 Definition

The set of all distinct terms of a sequence is called its **range**.

2.3 Definition

A sequence $\{x_n\}$ is **bounded** if its range is bounded.

Hence $\{x_n\}$ is bounded, if there exist reals k', k such that $k' \le x_n \le k$, $\forall\ n \in N$ or equivalently if there exist $K \ge 0$ such that $|x_n| \le K$, $\forall\ n \in N$.

2.4 Definition

A sequence is said to be **unbounded** if it is not bounded.

2.5 Theorem

Limit of a sequence, if it exists, is unique.

2.6 Theorem

Every convergent sequence is bounded but the converse is not true.

The following example shows that the converse of the theorem is not true.

2.7 Example

Let the sequence $\{x_n\}$ defined by $x_n = (-1)^n$.

The sequence has two cluster points 1 and -1, and the sequence is bounded because its range set is $\{1, -1\}$. But the sequence is not convergent.

2.8 Definition

A sequence $\{x_n\}$ is said to be **monotonically increasing** if $x_{n+1} \ge x_n$, $\forall\ n \in N$ and **monotonically decreasing** if $x_{n+1} \le x_n$, $\forall\ n \in N$.

2.9 Theorem

Suppose $\{x_n\}$ is monotonic. Then $\{x_n\}$ converges if and only if it is bounded.

Proof:

Suppose that $\{x_n\}$ is a monotonically increasing sequence.

Then $x_n \le x_{n+1}$, for each n. Let E be the range of $\{x_n\}$.

Suppose that $\{x_n\}$ is bounded, and x is the supremum of E.

Then $x_n \le x$, for each n. Since $\sup E = x$, there is an integer N such that $x - \varepsilon < x_N \le x$, $\forall\ \varepsilon > 0$.

Since $\{x_n\}$ is increasing, $x - \varepsilon < x_n \le x$, for $n \ge N$ and which shows that $\{x_n\}$ converges to x.

By Theorem 2.6, the converse is true. □

2.10 Theorem

(i) If $p > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$.

(ii) If $n > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

(iii) If $p > 0$, and α is real, then $\lim\limits_{n\to\infty} \dfrac{n^{\alpha}}{(1+p)^{n}} = 0$.

(iv) If $|x| < 1$, then $\lim\limits_{n\to\infty} x^{n} = 0$.

Proof:

(i) Take any $\varepsilon > 0$.

Choose an integer N such that $N > \left(\dfrac{1}{\varepsilon}\right)^{1/p}$.

Then $N^{p} > \dfrac{1}{\varepsilon}$. Then $n \geq N$ implies $\left|\dfrac{1}{n^{p}} - 0\right| \leq \left|\dfrac{1}{N^{p}}\right| < \varepsilon,\ \forall\ \varepsilon > 0$. □

(ii) If $p > 1$ and let $x_{n} = \sqrt[n]{p} - 1$, then $x_{n} > 0$ and by the binomial theorem,

$1 + nx_{n} \leq \left(1 + x_{n}\right)^{n} = p$.

Thus $0 < x_{n} \leq \dfrac{p-1}{n}$.

Since $\dfrac{p-1}{n} \to 0$ as $n \to \infty$, $x_{n} \to 0$ as $n \to \infty$.

If $p = 1$, then $x_{n} = 0$.

Thus $x_{n} \to 0$ as $n \to \infty$.

If $0 < p < 1$, then the result is obtained by choosing $x_{n} = \dfrac{1}{\sqrt[n]{p}} - 1$. □

(iii) Take $x_{n} = \sqrt[n]{n} - 1$.

Then $x_{n} \geq 0$ and by the binomial theorem, $n = \left(1 + x_{n}\right)^{n} \geq \dfrac{n(n-1)}{2}\, x_{n}^{2}$.

Then $0 \leq x_{n} \leq \sqrt{\dfrac{2}{n-1}}$ for $n \geq 2$.

Hence $x_{n} \to 0$ as $n \to \infty$. □

(iv) Let k be an integer such that $k > \alpha$ and $k > 0$. For $n > 2k$,

$$\left(1+p\right)^{n} > \frac{n(n-1)\cdots(n-k+1)}{k!}\, p^{k} > \frac{n^{k}p^{k}}{2^{k}k!}.$$

Then $0 < \dfrac{1}{(1+p)^{n}} < \dfrac{2^{k}k!}{n^{k}p^{k}}$.

Thus $0 < \dfrac{n^{\alpha}}{(1+p)^{n}} < \dfrac{2^{k}k!}{n^{k}p^{k}}\, n^{\alpha}$.

Since $k > \alpha$, $\alpha - k < 0$ and $\dfrac{1}{n^{k-\alpha}} \to 0$ as $n \to \infty$. □

(v) If $x = 0$, it is obvious.

If $x \neq 0$ and $|x| < 1$, by (iv) we choose $p = \dfrac{1}{x} - 1$, then $p > 0$ and α is real.

Take $\alpha = 0$. We get $\lim\limits_{n\to\infty} \frac{1}{(1+p)^n} = 0.$

3. Convergence of Some Infinite Series

3.1 Definition

Given a sequence $\{a_n\}$ and consider the sum $a_p + a_{p+1} + \cdots + a_q$, we use the notation $\sum\limits_{n=p}^{q} a_n$ $(p \le q)$.

With $\{a_n\}$ an associate sequence $\{s_n\}$ is chosen, where $s_n = \sum\limits_{k=1}^{n} a_k$ (**the partial sums of the series**).

For $\{s_n\}$, $\sum\limits_{n=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ is an infinite series or just a series. If $\{s_n\}$ converges to s the **series converges** and write $\sum\limits_{n=1}^{\infty} a_n = s$ (the sum of the series).

But it should be clearly understood that s is the limit of a sequence of sums, and is obtained simply by addition.

If $\{s_n\}$ diverges, the series **diverges** (does not converge).

Sometimes, we shall consider series of the form $\sum\limits_{n=0}^{\infty} a_n$, for convenience of notation. And frequently, we shall simply write $\sum a_n$ in place of $\sum\limits_{n=1}^{\infty} a_n$ or $\sum\limits_{n=0}^{\infty} a_n$.

It is clear that every theorem about sequences can be stated in terms of series (letting $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n > 1$) and vice versa. But it is nevertheless useful to consider both concepts.

3.2 Examples

(i) $\sum \frac{1}{n}$ diverges.

For the given series, $u_n = \frac{1}{n}$.

So $s_1 = 1$, $s_2 = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$, $s_3 = 1 + \frac{1}{2} + \frac{1}{3} = 1.533$,

$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1.783$, $\cdots$, $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and so on.

Then the partial sums s_n of the series are tends to ∞ as $n \to \infty$.

(ii) $\sum \frac{1}{n(n+1)}$ converges.

For the given series, $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$s_1 = 1 - \frac{1}{2}$,

$s_2 = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) = 1-\frac{1}{3}$,

$s_3 = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) = 1-\frac{1}{4}$, $\cdots$,

$s_n = 1-\frac{1}{n+1}$ and so on.

Then $\lim_{n\to\infty} s_n = 1$.

Thus the partial sums of the series converges to 1.

Hence the given series converges and its sum is 1.

3.3 Theorem

$\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that

$$\left|\sum_{k=n}^{m} a_k\right| \le \varepsilon \text{ if } m \ge n \ge N.$$

In particular, taking $m = n$, Equation (1) becomes $|a_n| \le \varepsilon$ $(n \ge N)$.

3.4 Theorem

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof:

Let $s_n = a_1 + a_2 + \cdots + a_n$ and $\lim_{n\to\infty} s_n = s$. Since $a_n = s_n - s_{n-1}$, $\lim_{n\to\infty} a_n = 0$. □

Converse of the theorem is not true. We will show it by the following example.

3.5 Example

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

For $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

But $s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \sqrt{n}$.

So $s_n \to \infty$ as $n \to \infty$.

Hence the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

3.6 Theorem

A series of non-negative terms converges if and only if its partial sums form a bounded sequence.

3.7 Theorem

(i) If $|a_n| \le c_n$ for $n \ge N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(ii) If $a_n \ge d_n \ge 0$ for $n \ge N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof:

(i) Suppose that $|a_n| \le c_n$ for $n \ge N_0$, where N_0 is some fixed integer and $\sum c_n$ converges.

By Theorem 3.3, given $\varepsilon > 0$, there exists an integer $N \ge N_0$ such that $\left|\sum_{k=n}^{m} c_k\right| \le \varepsilon$ if $m \ge n \ge N$.

Since $|a_n| \le c_n$ for $n \ge N_0$, $\sum_{k=n}^{m} |a_k| \le \sum_{k=n}^{m} c_k \le \varepsilon$.

Again $\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k| \le \varepsilon$.

By Theorem 3.3, $\sum a_n$ converges. $\square$

(ii) Since $0 \le d_n \le a_n$ for $n \ge N_0$, $|d_n| \le a_n$ for $n \ge N_0$.

By (i) if $\sum a_n$ converges then $\sum d_n$ converges.

Suppose that $\sum d_n$ diverges.

Then by Theorem 3.6, its partial sums does not form a bounded sequence. So the partial sum of $\{a_n\}$ cannot be a bounded sequence.

Then by Theorem 3.6, $\sum a_n$ diverges. $\square$

Now consider the convergence of two non-negative terms of the series $\sum x^n$ and $\sum \frac{1}{n^p}$ which will play an important part of the series.

3.8 Theorem

If $0 \le x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Proof:

If $x = 0$, then $\sum_{n=0}^{\infty} x^n = 1$.

If $0 < x < 1$ then $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ and its sum is $\frac{1}{1-x}$.

If $x = 1$, then $\sum_{n=0}^{\infty} x^n = 1 + 1 + 1 + \cdots$.

Hence $\sum_{n=0}^{\infty} x^n$ diverges if $x = 1$.

Similarly, if $x > 1$, the series $\sum_{n=0}^{\infty} x^n$ diverges. $\square$

3.9 Theorem

$\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Proof:

Let $p > 1$.

We have $2^n > n,\ \forall\ n \in N$. Since s_n is the partial sum of the given series and the terms of the series are all non-negative, $s_n < s_{2^n}$.

$$s_{2^n} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{(2^n)^p} \text{ and}$$

$$\begin{aligned} s_{2^{n+1}-1} &= \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{(2^{n+1}-1)^p} \\ &= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) \\ &\quad + \cdots + \left(\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \cdots + \frac{1}{(2^{n+1}-1)^p}\right) \end{aligned}$$

$$\begin{aligned} s_{2^{n+1}-1} &< \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \cdots + \frac{2^n}{(2^n)^p} \\ &= \frac{1}{1^p} + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \cdots + \frac{1}{2^{n(p-1)}}. \end{aligned}$$

Thus on the right side is of the form of a geometric series and its sum is $\dfrac{1-\left(\frac{1}{2^{p-1}}\right)^{n+1}}{1-\frac{1}{2^{p-1}}}$.

So $s_{2^{n+1}-1} < \dfrac{1-\left(\frac{1}{2^{p-1}}\right)^{n+1}}{1-\frac{1}{2^{p-1}}} < \dfrac{2^{p-1}}{2^{p-1}-1},\ \forall\ n \in N.$

Since $2^n < 2^{n+1}-1,\ 0 \le s_{2^n} < s_{2^{n+1}-1},\ \forall\ n \in N.$

Thus the partial sums of the given series form a bounded sequence.

By Theorem 3.6, $\sum \frac{1}{n^p}$ converges if $p > 1$.

Now, take $p = 1$. Then the given series is $\sum \frac{1}{n}$.

Consider $s_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n}$

$$\begin{aligned} &= \left(1+\frac{1}{2}\right) + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \ldots + \frac{1}{2^n}\right) \\ &> \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \ldots + \frac{2^{n-1}}{2^n} = \frac{n}{2}. \end{aligned}$$

By taking n sufficiently large enough, the partial sums of the series are not bounded above.

By Theorem 3.6, $\sum \frac{1}{n^p}$ diverges if $p = 1$.

Next, put $p < 1$. Then $n^p < n,\ \forall\ n \in N.$

So $\sum \frac{1}{n^p} > \sum \frac{1}{n},\ \forall\ n \in N.$

From the above result, $\sum \frac{1}{n}$ diverges and by Theorem 3.7(ii), $\sum \frac{1}{n^p}$ diverges if $p < 1$. □

4. Result and Discussion

This result has 3 facts. The first is that a series of non-negative terms will converge if the partial sums of the series form a bounded sequence and vice-versa. The second is that the series $\sum_{n=0}^{\infty} x^n$ converges if $0 < x < 1$. The last is that the series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Conclusion

Most of the people has been known the sum of the finite numbers of non-negative (positive) terms is an accurate number (finite number). From this research, the existence and uniqueness of the sum of the non-negative terms of the (infinite) series will be seen.

Acknowledgements

I would like to express my heartfelt gratitude to Dr. Tint Moe Thuzar, Rector, Yadanabon University, Dr. U Khin Myot, Dr. Myint Myint Oo and Dr. Khin Maw Maw Soe Pro-Rectors, Yadanabon University, for their permission to carry out the research and their encouragement. And then, I would like to thank Dr. Mon Yee Aye, Professor and Head of the Department of Mathematics, Dr. Daw Cho, Professor, Department of Mathematics, Yadanabon University, for their exhortation and helpful comments on this research.

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